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A c_0 -SATURATED BANACH SPACE WITH NO LONG UNCONDITIONAL BASIC SEQUENCES

J. LOPEZ-ABAD AND S. TODORCEVIC

ABSTRACT. We present a Banach space \mathfrak{X} with a Schauder basis of length ω_1 which is saturated by copies of c_0 and such that for every closed decomposition of a closed subspace $X = X_0 \oplus X_1$, either X_0 or X_1 has to be separable. This can be considered as the non-separable counterpart of the notion of hereditarily indecomposable space. Indeed, the subspaces of \mathfrak{X} have “few operators” in the sense that every bounded operator $T : X \rightarrow \mathfrak{X}$ from a subspace X of \mathfrak{X} into \mathfrak{X} is the sum of a multiple of the inclusion and a ω_1 -singular operator, i.e., an operator S which is not an isomorphism on any non-separable subspace of X . We also show that while \mathfrak{X} is not distortable (being c_0 -saturated), it is arbitrarily ω_1 -distortable in the sense that for every $\lambda > 1$ there is an equivalent norm $\|\cdot\|$ on \mathfrak{X} such that for every non-separable subspace X of \mathfrak{X} there exist $x, y \in S_X$ such that $\|x\|/\|y\| \geq \lambda$.

1. INTRODUCTION

The solutions of the unconditional basic sequence problem [8] and the distortion problem ([6], [11]) have an intricate connection. They both have profited on one side from the development of the Tsirelson-like constructions of conditional norms and on the other from the development of the infinite-dimensional Ramsey theory. These connections are however well understood only at the level of separable spaces. This paper, which can be considered as a natural continuation of our previous paper [1], is an attempt to explain these connections in the non-separable context as well. As byproducts we discover some new non-separable as well as separable phenomena. Our results are all based on a method of constructing Banach spaces with long Schauder bases of length ω_1 , a method which among other things crucially uses the information about the way the classical Ramsey theorem [13] fails in the uncountable context [15].

Recall that an infinite dimensional Banach space X is *indecomposable* if for every closed decomposition $X = X_0 \oplus X_1$ one of the subspaces X_0 or X_1 must be finite dimensional. The space X is *hereditary indecomposable*, HI in short, if every closed infinite dimensional subspace of E is indecomposable. The first example of such a space was constructed by Gowers-Maurey [8] as a byproduct of their solution of the unconditional basic sequence problem. The paper [1] considers the unconditional basic sequence problem in the context of Banach spaces that are not necessarily separable. In particular [1] produces a reflexive Banach space \mathfrak{X}_{ω_1} with a Schauder basis $(e_\alpha)_{\alpha < \omega_1}$ of length ω_1 with no infinite unconditional basic sequence. Applying Gowers' dichotomy [7], one concludes that, while \mathfrak{X}_{ω_1} is decomposable (as for example $\mathfrak{X}_{\omega_1} = \overline{\langle e_\alpha \rangle_{\alpha < \omega}} \oplus \overline{\langle e_\alpha \rangle_{\alpha \geq \omega}}$) it is saturated by hereditarily indecomposable subspaces (HI-*saturated* in short), i.e., every closed infinite dimensional subspace of \mathfrak{X}_{ω_1} contains an hereditarily indecomposable Banach space.

In this paper we present an example of a Banach space \mathfrak{X} with a Schauder basis of length ω_1 with some extreme discrepancies between properties of the class of separable subspaces of \mathfrak{X} and the class of non-separable subspaces of \mathfrak{X} . For example, at the separable level, \mathfrak{X} is *saturated* by copies of c_0 , i.e. every closed infinite dimensional subspace of \mathfrak{X} contains an isomorphic copy of c_0 . So, in particular, every closed infinite dimensional subspace contains an infinite unconditional basic sequence. On the non-separable level, \mathfrak{X} contains no unconditional basic sequence of length ω_1 . More precisely, for every closed subspace X of \mathfrak{X} and every decomposition $X = X_0 \oplus X_1$ one of the spaces X_0 or X_1 must be separable. In fact every bounded operator $T : X \rightarrow \mathfrak{X}$, where X is a closed subspace of \mathfrak{X} , can be decomposed as

$$T = \lambda i_{X, \mathfrak{X}} + S,$$

where $i_{X, \mathfrak{X}} : X \rightarrow \mathfrak{X}$ is the inclusion mapping, and S is an ω_1 -singular operator, the non-separable counterpart of the notion of *strictly singular operator*, which requires that S is not an isomorphism on any non-separable subspace of X .

Another discrepancy between the behavior of separable subspaces of \mathfrak{X} and non-separable ones (as well as a striking distinction between \mathfrak{X} and \mathfrak{X}_{ω_1}) comes when one considers the distortion constants of its equivalent norms. Recall that the *distortion* of an equivalent norm $||| \cdot |||$ of a Banach space $(X, \| \cdot \|)$ is the constant

$$d(X, ||| \cdot |||) = \inf_Y \sup \left\{ \frac{|||x|||}{|||y|||} : x, y \in S_{(Y, ||| \cdot |||)} \right\}$$

where the infimum is taken over all *infinite dimensional* subspaces Y of X . One says that $(X, \| \cdot \|)$ is *arbitrarily distortable* if

$$\sup_{||| \cdot |||} d(X, ||| \cdot |||) = \infty$$

where the supremum is taken over all equivalent norms $||| \cdot |||$ of $(X, \| \cdot \|)$. The analysis of our space \mathfrak{X} suggests the following variation of this notion of distortion. Given an equivalent norm $||| \cdot |||$ of a Banach space $(X, \| \cdot \|)$, let

$$d_{\omega_1}(X, ||| \cdot |||) = \inf_Y \sup \left\{ \frac{|||x|||}{|||y|||} : x, y \in S_{(Y, ||| \cdot |||)} \right\}$$

where the infimum now is taken over all *non-separable* subspaces Y of X . We say that $(X, \| \cdot \|)$ is *arbitrarily ω_1 -distortable* when

$$\sup_{||| \cdot |||} d_{\omega_1}(X, ||| \cdot |||) = \infty$$

where again the supremum is taken over all equivalent norms $||| \cdot |||$ of $(X, \| \cdot \|)$. It has been shown in [1] that the space \mathfrak{X}_{ω_1} is arbitrarily distortable. Note however that our space \mathfrak{X} is *not* distortable at all, i.e. $d(\mathfrak{X}, ||| \cdot |||) = 1$ for every equivalent norm $||| \cdot |||$ on \mathfrak{X} . This is a consequence of the fact that \mathfrak{X} is c_0 -saturated and the well-known result of R. C. James [9] which states that if a Banach space contains isomorphic copies of c_0 then it contains almost isometric copies of c_0 . Nevertheless, it turns out that the space \mathfrak{X} is distortable in the non-separable sense, i.e., \mathfrak{X} is arbitrarily ω_1 -distortable. It follows that, while the arbitrary distortion of a Banach space X implies its arbitrary ω_1 -distortion, the converse implication is not true.

2. DEFINITION OF THE SPACE \mathfrak{X}

The construction of the space \mathfrak{X} relies on the construction of the Banach space \mathfrak{X}_{ω_1} from [1]. So, in order to avoid unnecessary repetitions, we assume the reader is familiar with standard definitions and results in this area (see for example [10] and [3]), and in particular with the way this has been amplified in [1] to the non-separable context.

The space \mathfrak{X} will be defined as the completion of $(c_{00}, \|\cdot\|)$ under the norm $\|\cdot\|$ induced by a set of functionals $K \subseteq c_{00}(\omega_1)$.

Definition 2.1. Let K be the minimal subset of $c_{00}(\omega_1)$ satisfying the following conditions:

- (i) It contains $(e_\gamma^*)_{\gamma < \omega_1}$, is symmetric (i.e., $\phi \in K$ implies $-\phi \in K$) and is closed under the restriction on intervals of ω_1 .
- (ii) For every *separated* block sequence $(\phi_i)_{i=1}^d \subseteq K$, with $d \leq n_{2j}$, one has that the combination $(1/m_{2j}) \sum_{i=1}^{n_{2j}} \phi_i \in K$.
- (iii) For every separated *special* sequence $(\phi_i)_{i=1}^d \subseteq K$ with $d \leq n_{2j+1}$ one has that $\phi = (1/m_{2j+1}) \sum_{i=1}^{n_{2j+1}} \phi_i$ is in K . The functional ϕ is called a *special functional*.
- (iv) It is rationally convex.

Whenever $\phi \in K$ is of the form $\phi = (1/m_j) \sum_{i < d} \phi_i$ given in (ii) or (iii) above we say that ϕ has a *weight* $w(\phi) = m_j$. Finally, the norm on $c_{00}(\omega_1)$ is defined as

$$\|x\| = \sup\{\phi(x) = \sum_{\alpha} \phi(\alpha) \cdot x(\alpha) : \phi \in K\}$$

and \mathfrak{X} is the completion of $(c_{00}(\omega_1), \|\cdot\|)$.

Before we discuss the new notion of separated sequence, let us give a list of direct consequences from the definition of \mathfrak{X} .

REMARK 2.2. (a) It is clear that the norming set K presented here is a subset of the one introduced in [1] for the Banach space $\mathfrak{X}_{\omega_1} = (\mathfrak{X}_{\omega_1}, \|\cdot\|_{\omega_1})$. So, it follows that for every $x \in c_{00}(\omega_1)$ one has that $\|x\| \leq \|x\|_{\omega_1}$. This fact will be used frequently in this paper.

(b) By the minimality of K , there is the following natural notion of *complexity* of every element ϕ of K : Either $\phi = \pm e_\alpha^*$, or ϕ is a rational convex combination $\phi = \sum_{i < k} r_i f_i$ of elements $(f_i)_{i < k}$ of K , or $\phi = (1/m_j) \sum_{i < d} f_i$ for a separated block sequence $(f_i)_{i < d}$ in K with $d \leq n_j$. And in this latter case we say that $w(\phi) = m_j$ is a *weight* of ϕ .

(c) The property (i) makes the natural Hamel basis $(e_\alpha)_{\alpha < \omega_1}$ of $c_{00}(\omega_1)$ a *transfinite bimonotone Schauder basis* of \mathfrak{X} , i.e. $(e_\alpha)_{\alpha < \omega_1}$ is total and for every interval $I \subseteq \omega_1$ the corresponding projection $P_I : \mathfrak{X} \rightarrow \mathfrak{X}_I = \overline{\langle e_\alpha \rangle_{\alpha \in I}}$ has norm 1. Let us set $P_\gamma = P_{[0, \gamma]}$ and $\mathfrak{X}_\gamma = \mathfrak{X}_{[0, \gamma]}$ for every countable ordinal γ . It follows that every closed infinite dimensional subspace contains a further subspace isomorphic to the closed linear span of a block sequence of the basis (e_α) (see Proposition 1.3 in [1] for full details). This goes in contrast with the corresponding property of a Banach space with a Schauder basis $(x_k)_{k \in \mathbb{N}}$ for which it is well-known that every closed infinite dimensional subspace contains almost isometric copies of the closed linear span of a certain block sequence.

(d) The second property (ii) is responsible of the existence of semi-normalized averages in the span of every uncountable block sequence of \mathfrak{X} . The third (iii) and fourth (iv) properties makes

every operator from a closed subspace of \mathfrak{X} into \mathfrak{X} a multiple of the identity plus a ω_1 -singular operator.

(e) The basis $(e_\alpha)_{\alpha < \omega_1}$ is shrinking, i.e. $(e_{\alpha_n})_n$ is shrinking in the usual sense for every increasing sequence $(\alpha_n)_n$ of countable ordinals (the proof is essentially equal to that for the space \mathfrak{X}_{ω_1} provided in [1]; we leave the details to the reader). It follows that $(e_\alpha)_{\alpha < \omega_1}$ is an uncountable weakly-null sequence, i.e. for every $x^* \in \mathfrak{X}^*$ the numerical sequence $(x^*(e_\alpha))_{\alpha < \omega_1} \in c_0(\omega_1)$. This last property readily implies the following.

(f) Suppose that $T : \mathfrak{X} \rightarrow \mathfrak{X}$ is a bounded operator. Then for every uncountable subset A of ω_1 and every countable ordinal γ one has that $P_\gamma(T(e_\alpha)) = 0$ for all but countably many $\alpha \in A$.

Fix from now on a function $\varrho : [\omega_1]^2 \rightarrow \omega$ with the following properties:

- (i) $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.
- (ii) $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.
- (iii) $\{\alpha < \beta : \varrho(\alpha, \beta) \leq n\}$ is finite for all $\beta < \omega_1$ and $n \in \omega$.

The reader is referred to [15] and [16] for full discussion of this notion and constructions of various ϱ -functions. We shall use it here to measure “distances” between various subsets of ω_1 .

Definition 2.3. For two subsets s and t of ω_1 , set

$$\varrho(s, t) = \min\{\varrho(\alpha, \beta) : \alpha \in s, \beta \in t\}.$$

Given an integer p we say that s and t are p -separated if $\varrho(s, t) \geq p$. A sequence $(s_i)_i$ is p -separated if it is pairwise p -separated, i.e. s_i and s_j are p -separated for every $i \neq j$. The sequence $(s_i)_i$ is *separated* if it is $|\bigcup_i s_i|$ -separated.

Every notion we introduced here for sets of ordinals can be naturally transferred, via their supports, to vectors of \mathfrak{X} .

The following is a simple, but useful statement, will give use separated subsequences of any sufficiently long sequence of finite sets. It is the kind of result eventually used in showing that the norm of \mathfrak{X} keeps a substantial conditional structure when restricted on an arbitrary non-separable subspace of \mathfrak{X} .

Proposition 2.4. *Let $(A_i)_{i < n}$ be a block sequence of subsets of ω_1 , each of them of order-type ω . Then for every block sequence $(s_\alpha)_{\alpha \in \bigcup_{i < n} A_i}$ of finite sets of countable ordinals and every integer p there are $\alpha_i \in A_i$ ($i < n$) such that $(s_{\alpha_i})_{i < n}$ is p -separated.*

PROOF. This is done by induction on n . Fix all data as in the statement for $n > 1$. Then let $\alpha_{n-1} = \min A_{n-1}$. Since the set

$$\overline{s_{\alpha_{n-1}}}^p = \{\beta < \omega_1 : \text{there is some } \gamma \in s_{\alpha_{n-1}} \text{ with } \beta \leq \gamma \text{ and } \varrho(\beta, \gamma) \leq p\}$$

is, by property (iii) of ϱ , finite, one easily obtains infinite subsets B_i of A_i ($i < n-1$) such that $s_\alpha \cap \overline{s_{\alpha_{n-1}}}^p = \emptyset$ for every $\alpha \in B_i$ and all $i < n-1$. By inductive hypothesis, there are $\alpha_i \in B_i$ ($i < n-1$) such that $(s_{\alpha_i})_{i < n-1}$ is p -separated. Obviously $(s_{\alpha_i})_{i < n}$ is the desired p -separated sequence. \square

Corollary 2.5. *Let n be an integer and let $(s_\alpha^i)_{\alpha < \omega_1}$ be a block sequence of finite set of countable ordinals for every $i < n$. Then there are $\alpha_0 < \dots < \alpha_{n-1}$ such that $(s_{\alpha_i}^i)_{i < n}$ is a separated block sequence.*

PROOF. Let A be an uncountable set such that $|s_\alpha^i| = p_i$ for every $i \in A$ and every $i < n$. Now for each $i < n$, let $A_i \subseteq A$ be of order type ω and such that $A_i < A_j$ and $s_\alpha^i < s_\beta^j$ if $i < j < n$ and $\alpha \in A_i, \beta \in A_j$ are such that $\alpha < \beta$. Then apply the previous proposition to $(t_\alpha)_{\alpha \in A_0 \cup \dots \cup A_{n-1}}$ and $\sum_{i < n} p_i$, where $t_\alpha = s_\alpha^i$ for the unique $i < n$ such that $\alpha \in A_i$. \square

2.1. Rapidly increasing sequences and deciding pairs. We now introduce some standard technical tools in this field. Particularly, a sort of vectors, called ℓ_1^k -averages, and the so-called rapidly increasing sequences (RIS in short). The importance of rapidly increasing sequences (x_n) is that it is possible to estimate the norm of linear combinations of (x_n) in terms of norms of linear combinations of the basis $(e_\alpha)_{\alpha < \omega_1}$. In this sense RIS behave like a subsequence $(e_{\alpha_k})_k$ of basis $(e_\alpha)_{\alpha < \omega_1}$. The role of ℓ_1^k -averages is that they are useful in creating RIS. We give the precise definitions now.

Definition 2.6. Let $C, \varepsilon > 0$. A normalized block sequence $(x_k)_k$ of X is called a (C, ε) -rapidly increasing sequence $((C, \varepsilon)$ -RIS in short) iff there is an increasing sequence $(j_k)_k$ of integers such that for all k ,

- (i) $\|x_k\| \leq C$,
- (ii) $|\text{supp } x_k| \leq m_{j_{k+1}} \varepsilon$ and
- (iii) For every functional $\phi \in K$ of type I with $w(\phi) < m_{j_k}$ one has that $|\phi(x_k)| \leq C/w(\phi)$.

Let $C > 0$ and $k \in \mathbb{N}$. A normalized vector y is called a $C - \ell_1^k$ -average iff there is a finite block sequence (x_0, \dots, x_{k-1}) such that $y = (x_0 + \dots + x_{k-1})/k$ and $\|x_i\| \leq C$.

First observe that any ω -subsequence $(e_{\alpha_n})_n$ of the basis is a $(1, \varepsilon)$ -RIS for every ε . Note also that it follows easily from the definition that if (x_n) is a (C, ε) -RIS, then for every $\varepsilon' > 0$ there is a subsequence $(x_n)_{n \in A}$ of (x_n) which is a (C, ε') -RIS.

As the norming set K we are using here is not saturated by “free” combinations of the form $(1/m_{2j}) \sum_{i < n_{2j}} f_i$, one cannot expect that there are ℓ_1^k -averages in the span of arbitrary block sequence. Indeed, as Theorem 3.2 shows, this is not the case for most of the block sequences, since clearly if (x_n) is C -equivalent to the c_0 -basis, then $\|(x_0 + \dots + x_k)/k\| \leq C/k$ for every k . However, the next Proposition guarantees their existence. Its proof is the natural modification of the standard proof for the separable case, which can be found, for example, in [2] or in [8].

Proposition 2.7. *For every $k \in \mathbb{N}$ there is $l = l(k) \in \mathbb{N}$ such that the following holds: Suppose that $(x_i)_{i < l}$ is a normalized block sequence with the property that there exists a separated sequence $(\phi_i)_{i < l}$ which is biorthogonal to $(x_i)_{i < l}$. Then $\langle x_i \rangle_{i < l}$ contains $2\text{-}\ell_1^k$ -averages.* \square

We recall the following is also a well known fact about ℓ_1^k -averages, connecting them to RIS.

Proposition 2.8. *Suppose that y is a $C - \ell_1^k$ -average and suppose that $E_0 < \dots < E_{l-1}$ are intervals with $n < k$. Then $\sum_{i=0}^{l-1} \|E_i y\| \leq C(1 + 2l/k)$. As a consequence, if y is a $C - \ell_1^{n_j}$ -average and $\phi \in K$ is with $w(\phi) < m_j$, then $|\phi(y)| \leq 3C/2w(\phi)$.*

In particular, for $2 - \ell_1^{n_j}$ -averages we get that $|\phi(y)| \leq 3/w(\phi)$ if $w(\phi) < m_j$. \square

REMARK 2.9. It follows that if $\mathbf{x} = (x_k)_k$ is a normalized block sequence such that each x_k is a $2 - \ell_1^{n_{j_k}}$ -average with $|\text{supp } x_k| \leq m_{j_k+1}$, then \mathbf{x} is a $(3, 1)$ -RIS. It readily follows that if $(x_k)_k$ is a normalized block sequence having a separated biorthogonal pair, then the linear span of $(x_k)_k$ contains $(3, \varepsilon)$ -RIS for every $\varepsilon > 0$.

Let $<_{\text{antilex}}$ denote the anti-lexicographical ordering on $\mathbb{N} \times \omega_1$. Whenever we say that a sequence $(x_k^\alpha)_{(k,\alpha) \in A}$ indexed on a subset A of $\mathbb{N} \times \omega_1$ is a block sequence we mean that x_k^α is finitely supported and that $x_{k_0}^{\alpha_0} < x_{k_1}^{\alpha_1}$ whenever $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$. In the following definitions and lemma, we introduce two genuinely non-separable tools. They are necessary for us because it not true in general that a non-separable subspace of \mathfrak{X} contains an almost isometric copy of the closed linear span of some uncountable block sequence. Recall that if X is a separable Banach space with a Schauder basis of length ω , the corresponding result is true and very frequently used.

Definition 2.10. Let $C, \varepsilon > 0$. We call a sequence $\mathbf{x} = (x_k^\alpha)_{(k,\alpha) \in \mathbb{N} \times \omega_1}$ a *long rapidly increasing sequence* (LRIS in short) iff

- (i) $x_{k_0}^{\alpha_0} < x_{k_1}^{\alpha_1}$ for every $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$, and
- (ii) The cardinality of $\text{supp } x_k^\alpha$ only depends on k .
- (iii) There is a sequence of integers (j_k) such that x_k^α is a $2 - \ell_1^{n_{j_k}}$ -average and $|\text{supp } x_k^\alpha| < m_{j_k+1}$ for every (k, α) .

REMARK 2.11. The chosen name is because it follows from the definition and Remark 2.9 that if $f = (f_0, f_1) : \mathbb{N} \rightarrow \mathbb{N} \times \omega_1$ is such that

$$\text{if } k < l \text{ then } f_0(k) < f_0(l) \text{ and } f_1(k) \leq f_1(l)$$

then one has that $(x_{f_0(k)}^{f_1(k)})_{k \in \mathbb{N}}$ is a $(3, 1)$ -RIS.

Definition 2.12. Given $\varepsilon > 0$ and two vectors $x, y \in c_0(\omega_1)$, with $x \neq 0$, we write $x <_\varepsilon y$ to denote that

$$\|P_{\text{supp } x} y\| < \varepsilon.$$

By technical reasons, we declare $0 <_\varepsilon y$ for every y .

We recall that for a vector $x \in c_0(\omega_1)$, one sets $\text{ran } x = [\min \text{supp } x, \sup \text{supp } x]$.

Definition 2.13. Given a bounded operator $T : X \rightarrow \mathfrak{X}$, where X is a closed non-separable subspace of \mathfrak{X} , we say that the couple (\mathbf{x}, \mathbf{y}) is a *deciding pair* for T if the following holds:

- (i) $\mathbf{x} = (x_k^\alpha)_{(k,\alpha) \in \mathbb{N} \times \omega_1} \subseteq X$ and $\mathbf{y} = (y_k^\alpha)_{(k,\alpha) \in \mathbb{N} \times \omega_1}$ is a LRIS and $\mathbf{x} \subseteq X$.
- (ii) $x_k^\alpha \in X$ and $\text{ran } y_k^\alpha \subseteq \text{ran } x_k^\alpha$ for every pair (k, α) .
- (iii) $\sum_{k \in \mathbb{N}} \|x_k^\alpha - y_k^\alpha\| \leq 1$ for every α .
- (iv) $x_{k_0}^{\alpha_0} <_{\varepsilon_{k_1}} x_{k_1}^{\alpha_1}$ and $T(x_{k_0}^{\alpha_0}) <_{\varepsilon_{k_1}} T(x_{k_1}^{\alpha_1})$ for every $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$.

A *transversal subsequence* of a double-indexed sequence $(x_k^\alpha)_{(k,\alpha)}$ is a finite subsequence of the form $(x_{k_i}^{\alpha_i})_i$ where $k_i < k_{i+1}$ and $\alpha_i < \alpha_{i+1}$ for every i .

In other words a deciding pair is nothing else but an uncountable ordered sequence of rapidly increasing sequences $(y_k^\alpha)_{k \in \mathbb{N}}$ which are asymptotically closed to a sequence $(x_k^\alpha)_{k \in \mathbb{N}}$ in X for which the corresponding sequence of images $(T x_k^\alpha)_{k \in \mathbb{N}}$ is “almost” block ordered.

Before we prove that deciding pairs always exist, we give some explanation of this notion.

Proposition 2.14. *Suppose that (\mathbf{x}, \mathbf{y}) is a deciding pair for T . Then for every $\varepsilon > 0$ and every integer l there is a transversal subsequence $\mathbf{z} = (y_{k_i}^{\alpha_i})_{i < l}$ of \mathbf{y} such that*

(i) \mathbf{z} is a $(3, \varepsilon)$ -RIS.

(ii) \mathbf{z} has a biorthogonal separated block sequence in the norming set K .

(iii) $\sum_{i < l} \|x_{k_i}^{\alpha_i} - y_{k_i}^{\alpha_i}\| \leq \varepsilon$.

PROOF. Fix all data. Let $M \subseteq \mathbb{N}$ be such that for every $\alpha < \omega_1$ one has that $(y_k^\alpha)_{k \in M}$ is a $(3, \varepsilon)$ -RIS and such that $\sum_{k \in M} \|x_k^\alpha - y_k^\alpha\| \leq \varepsilon$. Fix also for each pair (k, α) a functional $\phi_k^\alpha \in K$ such that $\phi_k^\alpha(y_k^\alpha) = 1$ and with $\text{ran } \phi_k^\alpha \subseteq \text{ran } y_k^\alpha$. Now apply Corollary 2.5 to $(\text{supp } \phi_k^\alpha)_{\alpha < \omega_1}$ ($k \in M$) to find a transversal subsequence $(\phi_{k_i}^{\alpha_i})_{i < l}$. Then $(y_{k_i}^{\alpha_i})_{i < l}$ is the desired sequence. \square

Lemma 2.15. *Every bounded operator $T : X \rightarrow \mathfrak{X}$ with X non-separable has a deciding pair.*

PROOF. First we make the following approximation to the final result.

Claim 1. *For every integer k and every $\varepsilon > 0$ there are two normalized sequences $(x_\alpha)_{\alpha < \omega_1}$ and $(y_\alpha)_{\alpha < \omega_1}$ such that*

(a) $x_\alpha \in X$ ($\alpha < \omega_1$).

(b) $\text{ran } y_\alpha \subseteq \text{ran } x_\alpha$ and $\|x_\alpha - y_\alpha\| \leq \varepsilon$ ($\alpha < \omega_1$).

(c) y_α is a $2 - \ell_1^k$ -average and $|\text{supp } y_\alpha|$ is independent of $\alpha < \omega_1$.

(d) $x_\alpha <_\varepsilon x_\beta$ and $Tx_\alpha <_\varepsilon Ty_\alpha$ for every $\alpha < \beta$.

Let us show the desired result from this claim: Find recursively for each $k \in \mathbb{N}$ two sequences $(z_\alpha^k)_{\alpha < \omega_1}$ and $(t_\alpha^k)_{\alpha < \omega_1}$ as the result of the application of the previous claim to the integer n_{j_k} and ε_k and where j_k is chosen such that $|\text{supp } t_\alpha^{k-1}| < m_{j_k}$. Finally, it is not difficult to see that one can extract for every k a subsequence $(x_k^\alpha, y_k^\alpha)_{\alpha < \omega_1}$ of $(z_\alpha^k, t_\alpha^k)_{\alpha < \omega_1}$ with the property that $x_{k_0}^{\alpha_0} <_{\varepsilon_{k_1}} x_{k_1}^{\alpha_1}$ and $T(x_{k_0}^{\alpha_0}) <_{\varepsilon_{k_1}} T(x_{k_1}^{\alpha_1})$ for every $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$.

Let us give now a proof of the claim. Fix k and $\varepsilon > 0$, and let l be any integer given by Proposition 2.7 when applied to k . Set $\delta = \varepsilon/2l$. Now use that the bounded operator $U = U(T, \gamma) : X \rightarrow X_\gamma \oplus X_\gamma$ defined by $U(x) = (P_\gamma(x), P_\gamma(T(x)))$ is ω_1 -singular (because it has separable range) to find two normalized sequences $\mathbf{z} = (z_\alpha)_{\alpha < \omega_1}$ and $\mathbf{t} = (t_\alpha)_{\alpha < \omega_1}$, and a block sequence $(F_\alpha)_{\alpha < \omega_1}$ of finite sets $F_\alpha \subseteq \omega_1$ of size l such that

(e) $\mathbf{z} \subseteq X$, and \mathbf{t} is a block sequence.

(f) $\text{ran } t_\alpha \subseteq \text{ran } z_\alpha$ and $\|z_\alpha - t_\alpha\| \leq \delta$ for every α .

(g) $z_\alpha <_\delta z_\beta$ and $T(z_\alpha) <_\delta T(z_\beta)$ for every $\alpha < \beta$.

(h) For every countable ordinal α there is a separated block sequence $(f_\xi)_{\xi \in F_\alpha}$ that is biorthogonal to $(t_\xi)_{\xi \in F_\alpha}$.

We observe that (h) can be achieved by a simple application of Corollary 2.5. By Proposition 2.7, we can find a $2 - \ell_1^k$ -average $y_\alpha \in \langle t_\beta \rangle_{\beta \in F_\alpha}$ for each $\alpha < \omega_1$. It is easy to see that if x_α is an arbitrary normalized vector in $\langle z_\beta \rangle_{\beta \in F_\alpha}$ such that $\text{ran } y_\alpha \subseteq \text{ran } x_\alpha$ and $\|x_\alpha - y_\alpha\| \leq \varepsilon$ (and there is such vector), then the corresponding sequences $(x_\alpha)_{\alpha < \omega_1}$ and $(y_\alpha)_{\alpha < \omega_1}$ fulfill all the properties (a) to (d). \square

REMARK 2.16. It is easy to find for given two bounded operators $T_0, T_1 : X \rightarrow \mathfrak{X}$ a deciding pair (\mathbf{x}, \mathbf{y}) for, simultaneously, T_0 and T_1 . This can be done simply by replacing $U(T, \gamma)$ above by the mapping $\bar{U} : X \rightarrow \mathfrak{X}_\gamma^3$ defined by $x \mapsto \bar{U}(x) = (P_\gamma(x), P_\gamma(T_0(x)), P_\gamma(T_1(x)))$.

3. MAIN PROPERTIES OF THE SPACE \mathfrak{X}

3.1. c_0 -saturation. We are ready to prove that \mathfrak{X} is c_0 -saturated. We start with the following more informative result.

Lemma 3.1. *Suppose that (x_k) is a normalized block sequence such that*

$$\lim_{k \rightarrow \infty} \|x_k\|_\infty = 0.$$

Then (x_k) has a subsequence which is 5-equivalent to the natural basis of c_0 .

PROOF. Let $(y_k)_k$ be a subsequence of $(x_k)_k$ with $\|y_k\|_\infty \leq 1/(2^{k+2}k)$.

Claim 2. *There is an infinite set M of integers such that for every triple (k_0, k_1, k_2) in M one has that*

$$\max\{\varrho(\alpha, \beta) : \alpha \in \text{supp } y_{k_0}, \beta \in \text{supp } y_{k_1}\} < k_2. \quad (1)$$

Proof of Claim: We color every triple (k_0, k_1, k_2) of integers by 1 if (1) holds and 0 otherwise. By the classical Ramsey theorem we can find an infinite set M all whose triples are equally colored. If this color is 1, then we are done. Otherwise, suppose that this color is 0, and let us yield a contradiction. Fix two integers $k_0 < k_1$ in M . Then for every $k \in M$ with $k > k_1$ let $\alpha_k \in \text{supp } y_{k_0}$ and $\beta_k \in \text{supp } y_{k_1}$ with $\varrho(\alpha_k, \beta_k) \geq k$. Find an infinite $P \subseteq M$ with $(\alpha_k, \beta_k) = (\alpha, \beta)$ for every $k \in P$. Then $\varrho(\alpha, \beta) \geq k$ for every $k \in P$, a contradiction. \square

Fix such M from the claim. We show that $(y_n)_{n \in M}$ is 5-equivalent to the natural basis of c_0 . Observe that since the basis $(e_\alpha)_\alpha$ is bimonotone one has that $\|\sum_{n \in M} a_n y_n\| \geq \|(a_n)_{n \in M}\|_\infty$ for every sequence $(a_n)_{n \in M}$ of scalars. So it remains to show that $\|\sum_{n \in M} a_n y_n\| \leq 5\|(a_n)_{n \in M}\|_\infty$. This is done in the next.

Claim 3. *For every $\phi \in K$ and every sequence $(a_n)_{n \in M}$ of scalar one has that*

$$|\phi(\sum_n a_n y_n)| \leq 5\|(a_n)_{n \in M}\|_\infty. \quad (2)$$

Proof of Claim: Fix all data, and set $y = \sum_n a_n y_n$. The proof of (2) is done by induction on the complexity of ϕ . If $\phi = \pm e_\alpha^*$, the result is trivial. Suppose that ϕ is rational convex combination $\phi = \sum_{i < k} c_i f_i$ of elements $(f_i)_{i < n}$ of K . Then, by applying the inductive hypothesis to f_i 's, one has that $|\phi(y)| \leq \sum_{i < k} c_i |f_i(y)| \leq 5\|(a_n)\|_\infty$. Suppose now that $\phi = (1/m_j) \sum_{i < d} f_i$, where $d \leq n_j$ and $(f_i)_{i < d}$ is a separated block sequence in K . Let

$$\begin{aligned} a &= \{i < d : \text{supp } y \cap \text{supp } f_i \neq \emptyset\}, \text{ and for } i \in a \text{ let} \\ k(i) &= \min\{k : \text{supp } y_k \cap \text{supp } f_i \neq \emptyset\}, \text{ and} \\ L &= \{k(i) : i \in a\}. \end{aligned}$$

Claim 4. (a) *If $k < \min L$, then $\text{supp } y_k \cap \text{supp } f = \emptyset$.*

(b) *There is some $\bar{i} \in a$ such that for every $k > \max L$, then $\text{supp } y_k \cap \text{supp } f = \text{supp } y_k \cap \text{supp } f_{\bar{i}}$.*

(c) *For every two consecutive $k_0 < k_1$ in K there is some $\bar{i}(k_0, k_1) \in a$ such that for every k with $k_0 < k < k_1$ one has that $\text{supp } y_k \cap \text{supp } f = \text{supp } y_k \cap \text{supp } f_{\bar{i}(k_0, k_1)}$.*

Proof of Claim: The statement (a) is clear. It is not difficult to show that the statements (b) and (c) follow from the following, also non difficult, fact: Suppose that I is an interval of integers, and suppose that there are at least two integers i_0 and i_1 in a such that $\text{supp } f_{i_\varepsilon} \cap \bigcup_{k \in I} \text{supp } y_k \neq \emptyset$ for $\varepsilon = 0, 1$. Then $I \cap L \neq \emptyset$. \square

Now we consider two cases:

Case 1. The cardinality of L is at most two. If $L = \emptyset$, then $f(x) = 0$. Suppose that $L = \{\bar{k}\}$. Then using (a), (b) and the inductive hypothesis, above one obtains that

$$|f(x)| \leq \frac{1}{m_j} |f_{\bar{i}}(\sum_{k > \bar{k}} a_k y_k)| + |a_{\bar{k}}| \leq 2 \|(a_k)_k\|_\infty.$$

Finally suppose that $L = \{k_0, k_1\}$ with $k_0 < k_1$. Then one has that

$$|f(x)| \leq |a_{k_0}| + |a_{k_1}| + \frac{1}{m_j} \left| f_{\bar{i}(k_0, k_1)}(\sum_{k=k_0+1}^{k_1-1} a_k y_k) + f_{\bar{i}}(\sum_{k > k_1} a_k y_k) \right| \leq 5 \|(a_k)_k\|_\infty.$$

Case 2. The set L has cardinality at least three. Let $k_0 < k_1 < k_2$ be the least three elements of L . Find $i_0 < i_1$ in a such that $k(i_\varepsilon) = k_\varepsilon$ for $\varepsilon = 0, 1$, and then $\alpha_\varepsilon \in \text{supp } f_{i_\varepsilon} \cap \text{supp } y_{k_\varepsilon}$ for $\varepsilon = 0, 1$. It follows that

$$\|f\|_{\ell_1} \leq |\text{supp } f| \leq \varrho(\alpha_0, \alpha_1) < k_2.$$

Hence for every $k \geq k_2$ one has that

$$|f(y_k)| \leq \|f\|_{\ell_1} \|y_k\|_\infty \leq \frac{1}{2^{k+2}}. \quad (3)$$

Using the inequality (3), conditions (a)-(c) above and the inductive hypothesis applied to $f_{i(i_0, i_1)}$ and f_{i_1} one obtains that

$$\begin{aligned} |f(x)| &\leq |a_{k_0}| + |a_{k_1}| + \frac{1}{m_j} \left| f_{\bar{i}(k_0, k_1)}(\sum_{k=k_0+1}^{k_1-1} a_k y_k) + f_{\bar{i}(k_1, k_2)}(\sum_{k=k_1+1}^{k_2-1} a_k y_k) \right| + |f(\sum_{k \geq k_2} a_k y_k)| \leq \\ &\leq 2 \|(a_k)_k\|_\infty + \frac{5}{2} \|(a_k)_k\|_\infty + \frac{1}{2} \|(a_k)_k\|_\infty = 5 \|(a_n)_n\|_\infty, \end{aligned}$$

as desired. \square

\square

Theorem 3.2. \mathfrak{X} is c_0 -saturated.

PROOF. Fix an closed infinite dimensional subspace X of \mathfrak{X} . We may assume, by Remark 2.2 (c), that indeed X is the closed linear span of a normalized block sequence (x_k) . Suppose that c_0 does not embed isomorphically into X , and let us yield a contradiction. Consider the norm-one operator $\sum_k a_k x_k \in X \mapsto (a_k)_k \in c_0$. This is, by hypothesis, strictly singular. So we can find a normalized block subsequence $(y_k)_k$ of $(x_k)_k$ with $\lim_{k \rightarrow \infty} \|y_k\|_\infty = 0$. Then by Lemma 3.1 $(y_k)_k$ has a subsequence equivalent to the c_0 -basis, a contradiction. \square

3.2. Distortion. From the previous Theorem 3.2 one immediately obtains the following.

Corollary 3.3. \mathfrak{X} is not distortable. \square

In contrast to this, we have a strong distortion phenomenon in the level of non-separable subspaces of \mathfrak{X} :

Theorem 3.4. \mathfrak{X} is arbitrarily ω_1 -distortable.

PROOF. We follow some of the ideas used to show that \mathfrak{X}_{ω_1} is arbitrarily distortable (see Corollary 5.36 in [1]). For $j \in \mathbb{N}$, and $x \in \mathfrak{X}_{\omega_1}$, let

$$\|x\|_j = \sup\{\phi(x) : w(\phi) = m_{2j}\}.$$

Notice that, obviously, for every $\phi \in K$ one has that (ϕ) is a dependent sequence, hence $(1/m_{2j})\phi \in K$. It follows that $\|\cdot\|_j \leq \|\cdot\| \leq m_{2j}\|\cdot\|_j$. Fix a closed non-separable subspace X of \mathfrak{X} . Let (x, y) be a deciding pair for the inclusion mapping $i_{X, \mathfrak{X}} : X \rightarrow \mathfrak{X}$.

Now fix an integer l and $\varepsilon > 0$. Use Proposition 2.14 to find a transversal subsequence $z = (y_{k_i}^{\alpha_i})_{i < n_{2l}}$ of y such that

- (a) $\sum_{i < l} \|x_{k_i}^{\alpha_i} - y_{k_i}^{\alpha_i}\| \leq \varepsilon$, and
- (b) z is a $(3, n_{2l}^{-2})$ -RIS with a biorthogonal separated block sequence $(f_i)_{i < n_{2l}}$ in K . Set

$$z_l = \frac{m_{2l}}{n_{2l}} \sum_{i < l} y_{k_i}^{\alpha_i}, \phi_l = \frac{1}{m_{2l}} \sum_{i < l} f_i.$$

Then $\phi_l \in K$, and the pair (z_l, ϕ_l) is what we called in [1] (Definition 3.1, see also the proof of Proposition 4.12) a $(6, 2l)$ -exact pair. So, it follows that if $l = j$ then one has

$$1 \leq \|z_j\|_j \leq \|z_j\| \leq 6,$$

while if $l > j$ then

$$1 \leq \|z_l\| \leq 6 \text{ and } \|z_l\|_j \leq \frac{12}{m_{2j}}.$$

Hence for every $l > j$ one has the discrepancy

$$\frac{\|z_j/\|z_j\|\|_j}{\|z_l/\|z_l\|\|_j} \geq \frac{1/6}{12/m_{2j+1}} = \frac{m_{2j}}{72}. \quad (4)$$

Since the vectors z_k and z_l can be found to be arbitrary close to X , it follows that one can obtain a similar inequality to (4) for vectors in X . Hence $(\mathfrak{X}, \|\cdot\|)$ is arbitrarily ω_1 -distortable. \square

3.3. Operators. It turns out that while the space \mathfrak{X} is c_0 -saturated it has the following form of indecomposability at the non-separable level.

Definition 3.5. A Banach space X is ω_1 -indecomposable if for every decomposition $X = Y \oplus Z$ one has that either Y or Z is separable. We say that X is ω_1 -hereditarily indecomposable (ω_1 -HI in short) if every subspace of X is ω_1 -indecomposable.

This kind of indecomposability corresponds to the following notion of singularity for operators.

Definition 3.6. An operator $T : X \rightarrow Y$ is ω_1 -singular if T is not an isomorphism on any non-separable subspace of X .

Observe that strictly singular and separable range operators are ω_1 -singular. While the strictly singular operators and operators with separable ranges form closed ideals of the Banach algebra $\mathcal{L}(X)$ of all bounded operators from X into X , we do not know if, in general, this is also the case for the family $\mathcal{S}_{\omega_1}(X)$ of ω_1 -singular operators of X . Indeed we do not even know if $\mathcal{S}_{\omega_1}(X)$ is closed under sums. We shall show however that for subspaces X of our space \mathfrak{X} one does have the property that $\mathcal{S}_{\omega_1}(X)$ form a closed ideal in the algebra $\mathcal{L}(X)$.

Using this notion one can have the following sufficient condition for being ω_1 -HI.

Proposition 3.7. *Suppose that X has the property that for every subspace Y of X every bounded operator $T : Y \rightarrow X$ is of the form $T = \lambda i_{Y,X} + S$ where S is ω_1 -singular and $\lambda \in \mathbb{R}$. Then X is ω_1 -HI.*

PROOF. Otherwise, fix two nonseparable subspaces Y and Z of X such that $d(S_X, S_Y) > 0$. It follows that the two natural projections $P_Y : Y \oplus Z \rightarrow Y$ and $P_Z : Y \oplus Z \rightarrow Z$ are both bounded. Fix $\lambda \in \mathbb{R}$ such that $T = i_{Y,X} \circ P_Y = \lambda i_{Y \oplus Z, X} + S$ with S ω_1 -singular. Since $T^2 = T$, we have that

$$(\lambda^2 - \lambda)i_{Y \oplus Z, X} = ((1 - 2\lambda)i_{Y \oplus Z, X} - S) \circ S. \quad (5)$$

Since it is clear that $U \circ S$ is ω_1 -singular if S is ω_1 -singular, it follows from (5) that $\lambda^2 = \lambda$. Without loss of generality, we may assume that $\lambda = 1$ (if $\lambda = 0$ we replace Y by Z in the preceding argument). Since $P_Y|Z = 0$, we obtain that $S = -i_{Z,X}$, a contradiction. \square

REMARK 3.8. Recall that V. Ferenczi has shown in [5] that if a complex Banach spaces X is HI then every operator from a subspace Y of X into X is a multiple of the inclusion plus a strictly singular operator. We do not know if the analogous result is true for ω_1 -singular operators or, in other words, if the converse implication of Proposition 3.7 is true in the case of complex Banach spaces.

The main purpose of this subsection is the study of the operator space $\mathcal{L}(X, \mathfrak{X})$, where X is an arbitrary closed infinite dimensional subspace of \mathfrak{X} . For the next few lemmas we fix a bounded operator $T : X \rightarrow \mathfrak{X}$ from a closed infinite dimensional subspace X of \mathfrak{X} into the space \mathfrak{X} . We also fix a deciding pair (\mathbf{x}, \mathbf{y}) for T (see Definition 2.13).

Lemma 3.9. *For all but countably many $\alpha < \omega_1$ one has that $\lim_{k \rightarrow \infty} d(Tx_k^\alpha, \mathbb{R}x_k^\alpha) = 0$.*

PROOF. Otherwise, using the property (iii) of the deciding pair (\mathbf{x}, \mathbf{y}) and going to subsequences if necessary, we may assume that there is $\varepsilon > 0$ such that

$$\inf_{k \in \mathbb{N}} d(Tx_k^\alpha, \mathbb{R}y_k^\alpha) > \varepsilon$$

for every countable ordinal α . Now using Hahn-Banach theorem and the fact that the norming set K is closed under rational convex combinations and restrictions on intervals we can find for every pair $(k, \alpha) \in \mathbb{N} \times \omega_1$ a functional $f_k^\alpha \in K$ such that one has that

- (a) $(f_k^\alpha)_{(k, \alpha) \in \mathbb{N} \times \omega_1}$ is a block sequence and $\text{ran } f_k^\alpha \subseteq \text{ran } x_k^\alpha$ for every (k, α) , and
- (b) $|f_k^\alpha(y_k^\alpha)| \leq \varepsilon_k$ while $f_k^\alpha(T(x_k^\alpha)) \geq \varepsilon$ for every (k, α) .

Fix j with $\varepsilon_{m_{2j+1}} > 2\|T\|$. Now use Proposition 2.14 to find a sequence $(F_i^\alpha)_{(i, \alpha) \in n_{2j+1} \times \omega_1}$ of finite sets of pairs from $\mathbb{N} \times \omega_1$ such that

- (c) $|F_i^\alpha| = n_{2j_i}$ for every $(i, \alpha) \in n_{2j+1} \times \omega_1$,
- (d) $(y_k^\xi)_{(k, \xi) \in F_i^\alpha}$ is a $(3, (n_{2j_i})^{-2})$ -RIS and $(f_k^\xi)_{(k, \xi) \in F_i^\alpha}$ is a separated block sequence for every $(i, \alpha) \in n_{2j+1} \times \omega_1$.
- (e) $|f_k^\xi(T(y_k^\xi))|, \|x_k^\xi - y_k^\xi\| \leq n_{2j_i}^{-3}$ for every $(k, \xi) \in F_i^\alpha$ and every $(i, \alpha) \in n_{2j+1} \times \omega_1$.
- (f) $2j_i = \sigma_\varrho(\phi_0^\alpha, m_{2j_0}, p_0, \dots, \phi_{i-1}^\alpha, m_{2j_{i-1}}, p_{i-1})$ for every $(i, \alpha) \in n_{2j+1} \times \omega_1$ where p_i is an integer such that

$$p_i \geq \max\{p_0, \dots, p_{i-1}, n_{2j+1}^2, p_\varrho(\bigcup_{k < i} (\text{supp } \phi_k^\alpha \cup \text{supp } t_k^\alpha)), |\text{supp } t_{i-1}^\alpha| n_{2j+1}^2\},$$

and where

$$\begin{aligned} \phi_i^\alpha &= \frac{1}{m_{2j_i}} \sum_{(k, \xi) \in F_i^\alpha} f_k^\xi, \\ t_i^\alpha &= \frac{n_{2j_i}}{m_{2j_i}} \sum_{(k, \xi) \in F_i^\alpha} y_k^\xi. \end{aligned}$$

and σ_ϱ and p_ϱ are the coding and the ϱ -number, respectively, introduced in [1]. Set also

$$z_i^\alpha = \frac{n_{2j_i}}{m_{2j_i}} \sum_{(k, \xi) \in F_i^\alpha} x_k^\xi.$$

Now again using Corollary 2.5 we can find countable ordinals $\alpha_0 < \dots < \alpha_{n_{2j+1}-1}$ such that $(\phi_i^{\alpha_i})_{i < n_{2j+1}}$ is a separated block sequence. It follows that the sequence $((t_i^{\alpha_i}, \phi_i^{\alpha_i}))_{i < n_{2j+1}}$ is a (n_{2j+1}^{-2}, j) -dependent sequence, a slightly variation of the notion of $(0, j)$ -dependent sequence used in [1] (Definition 5.22; see also the proof of Proposition 5.24). The only change is that now one has that $|\phi_i^{\alpha_i}(t_i^{\alpha_i})| \leq 1/n_{2j+1}^2$ instead of zero. It follows that

$$\left\| \frac{1}{n_{2j+1}} \sum_{i < n_{2j+1}} t_i^{\alpha_i} \right\| \leq \frac{1}{m_{2j+1}^2}. \quad (6)$$

Hence, setting $z = (1/n_{2j+1}) \sum_{i < n_{2j+1}} z_i^{\alpha_i}$, one has that

$$\|z\| \leq \frac{2}{m_{2j+1}^2}. \quad (7)$$

By the other hand, since $\phi = (1/m_{2j+1}) \sum_{i < n_{2j+1}} \phi_i^{\alpha_i}$ is in K , it follows that

$$\|T(z)\| \geq \phi(T(z)) \geq \frac{\varepsilon}{m_{2j+1}}. \quad (8)$$

Putting (7) and (8) together one gets

$$\frac{\varepsilon}{m_{2j+1}} \leq \|T(z)\| \leq \|T\| \|z\| \leq \frac{2\|T\|}{m_{2j+1}^2},$$

and this is contradictory with the choice of j . \square

Now for each countable ordinal α , let $\lambda_k^\alpha = \lambda_k^\alpha(T, \mathbf{x}, \mathbf{y}) \in \mathbb{R}$ be such that

$$d(T(x_k^\alpha), \mathbb{R}x_k^\alpha) = \|T(x_k^\alpha) - \lambda_k^\alpha x_k^\alpha\|.$$

Lemma 3.10. *For all but countably many $\alpha < \omega_1$, the numerical sequence $(\lambda_k^\alpha)_k$ is convergent.*

PROOF. Otherwise, using Lemma 3.9, one can find two real numbers $\delta < \varepsilon$, an uncountable set $A \subseteq \omega_1$, for each $\alpha \in A$ two infinite disjoint subsets L_α and R_α of \mathbb{N} , and a block sequence $(f_k^\alpha)_{(k,\alpha) \in (L_\alpha \cup R_\alpha) \times A}$ in K such that

- (i) $\text{ran } f_k^\alpha \subseteq \text{ran } y_k^\alpha$, $f_k^\alpha(y_k^\alpha) = 1$ for every $(k, \alpha) \in (L_\alpha \cup R_\alpha) \times A$.
- (ii) For every $\alpha \in A$ one has that $f_k^\alpha(T(x_k^\alpha)) < \delta$ if $k \in L_\alpha$, and $f_k^\alpha(T(x_k^\alpha)) > \varepsilon$ if $k \in R_\alpha$.

Let $j \in \mathbb{N}$ be such that $(\varepsilon - \delta)m_{2j+1} \geq 4\|T\|$. We find, as in the proof of the previous Lemma 3.9 a sequence $(F_i^\alpha)_{(i,\alpha) \in n_{2j+1} \times \omega_1}$ of finite sets of pairs from $\mathbb{N} \times \omega_1$ such that (c), (d), and (f) as there holds, and also

- (e') For every $(k, \xi) \in F_i^\alpha$ one has that $\|x_k^\xi - y_k^\xi\| \leq n_{2j+1}^{-3}$ for every $(i, \alpha) \in n_{2j+1} \times \omega_1$, and $f_k^\xi(T(x_k^\xi)) \geq \varepsilon$ if i odd and $f_k^\xi(T(x_k^\xi)) \leq \delta$ if i even.

We set also ϕ_i^α , u_i^α and z_i^α as there. Let $\alpha_0 < \dots < \alpha_{n_{2j+1}-1}$ be such that $(\phi_i^{\alpha_i})_{i < n_{2j+1}}$ is a separated block sequence. It follows that the sequence $((t_i^{\alpha_i}, \phi_i^{\alpha_i}))_{i < n_{2j+1}}$ is a $(1, j)$ -dependent sequence (see Definition 3.3 in [1]), hence

$$\left\| \frac{1}{n_{2j+1}} \sum_{i < n_{2j+1}} (-1)^i u_i^{\alpha_i} \right\| \leq \frac{1}{m_{2j+1}^2}, \quad (9)$$

and so by the property (e') one has that

$$\|z\| \leq \frac{2}{m_{2j+1}^2}, \quad (10)$$

where $z = (1/n_{2j+1}) \sum_{i < n_{2j+1}} (-1)^i z_i^{\alpha_i}$. One also has, setting $\phi = (1/m_{2j+1}) \sum_{i < n_{2j+1}} \phi_i^{\alpha_i}$, that

$$\|T(z)\| \geq |\phi(Tz)| = \left| \frac{1}{m_{2j+1} n_{2j+1}} \sum_{i < n_{2j+1}} (-1)^i \phi_i^{\alpha_i}(T(z_i^{\alpha_i})) \right| \geq \frac{\varepsilon - \delta}{2m_{2j+1}}. \quad (11)$$

From (10) and (11) one easily gets a contradiction with the choice of j . \square

For every $\alpha < \omega_1$ let (if exists) $\lambda_\alpha = \lambda_\alpha(T, \mathbf{x}, \mathbf{y}) = \lim_{k \rightarrow \infty} \lambda_k^\alpha$.

Corollary 3.11. *There is a real number $\lambda = \lambda(T, \mathbf{x}, \mathbf{y})$ such that $\lambda_\alpha = \lambda$ for all but countably many α .*

PROOF. Otherwise, there are two reals $\delta < \varepsilon$ such that both sets $A_0 = \{\alpha < \omega_1 : \lambda_\alpha < \delta\}$ and $A_1 = \{\alpha < \omega_1 : \lambda_\alpha > \varepsilon\}$ are uncountable. Find for every $(k, \alpha) \in \mathbb{N} \times \omega_1$ a countable ordinals $\beta(k, \alpha)$ $((k, \alpha) \in \mathbb{N} \times \omega_1)$ such that

- (i) $\beta(k, \alpha) \in A_i$ if k is equal to $i \pmod{2}$.
- (ii) $((x_k^{\beta(k, \alpha)}, y_k^{\beta(k, \alpha)}))_{(k, \alpha) \in \mathbb{N} \times \omega_1}$ is a deciding pair (\mathbf{z}, \mathbf{t}) for T .

It follows that $(\lambda_k^\alpha(\mathbf{z}, \mathbf{t}))_k$ is never converging, contradicting Lemma 3.10. \square

Corollary 3.12. *The scalar $\lambda(T, \mathbf{x}, \mathbf{y})$ is independent of \mathbf{x} and \mathbf{y} . We call it $\lambda(T)$.*

PROOF. Fix two deciding pairs (\mathbf{x}, \mathbf{y}) and (\mathbf{z}, \mathbf{u}) for T . It is easy to define a third one (\mathbf{v}, \mathbf{w}) for T such that the sets

$$\begin{aligned} &\{\alpha < \omega_1 : (\exists \beta < \omega_1)(\forall k \in \mathbb{N}) (x_k^\alpha, y_k^\alpha) = (z_k^\beta, u_k^\beta)\} \\ &\{\alpha < \omega_1 : (\exists \beta < \omega_1)(\forall k \in \mathbb{N}) (v_k^\alpha, w_k^\alpha) = (z_k^\beta, u_k^\beta)\} \end{aligned}$$

are both uncountable. It follows that $\lambda(T, \mathbf{x}, \mathbf{y}) = \lambda(T, \mathbf{v}, \mathbf{w}) = \lambda(T, \mathbf{z}, \mathbf{u})$. \square

Theorem 3.13. *The mapping $\lambda : \mathcal{L}(X, \mathfrak{X}) \rightarrow \mathbb{R}$ which sends T to $\lambda(T)$ is a bounded linear functional whose kernel $\text{Ker}(\lambda)$ is equal to the family of all ω_1 -singular operators from X into \mathfrak{X} .*

PROOF. It is obvious from the definition of $\lambda(T)$ that $|\lambda(T)| \leq \|T\|$. We now show the linearity of λ . It is easy to see that $\lambda(\mu T) = \mu\lambda(T)$. Let us prove now that $\lambda(T_0 + T_1) = \lambda(T_0) + \lambda(T_1)$. Let (\mathbf{x}, \mathbf{y}) be a deciding pair for both T_0 and T_1 (See Remark 2.16). Let $\lambda_k^{\alpha,0}, \lambda_k^{\alpha,1}, \lambda_k^\alpha \in \mathbb{R}$ be such that $d((T_0 + T_1)(x_k^\alpha), \mathbb{R}x_k^\alpha) = \|(T_0 + T_1)(x_k^\alpha) - \lambda_k^\alpha x_k^\alpha\|$, and $d(T_i(x_k^\alpha), \mathbb{R}x_k^\alpha) = \|T(x_k^\alpha) - \lambda_k^{\alpha,i} x_k^\alpha\|$, for $i = 0, 1$. It follows from Lemma 3.9, applied to T_0, T_1 and $T_0 + T_1$, that

$$\lim_{k \rightarrow \infty} (\lambda_k^{\alpha,0} + \lambda_k^{\alpha,1}) - \lambda_k^\alpha = 0, \quad (12)$$

for all but countably many α . The desired result now follows from (12) and Lemma 3.10.

Now we prove that $\text{Ker}(\lambda)$ is the family of the ω_1 -singular operators. Suppose first that $\lambda(T) = 0$. We are going to show that T is not isomorphism when restricted to a non-separable subspace of X . To do this, let $\varepsilon > 0$, and let Z be a non-separable subspace of X . Let (\mathbf{x}, \mathbf{y}) be any deciding pair for T with $\mathbf{x} \subseteq Z$. Since $\lambda(T) = 0$, by previous Lemma 3.9 we can find $(k, \alpha) \in \mathbb{N} \times \omega_1$ such that $\|T(x_k^\alpha)\| < \varepsilon$, as desired.

Finally, suppose that T is a ω_1 -singular operator. Our intention is to provide a deciding pair (\mathbf{x}, \mathbf{y}) such that $\|T(x_k^\alpha)\| \leq 2^{-k}$ for every (k, α) . Then $\lambda_\alpha(T, \mathbf{x}, \mathbf{y}) = 0$ for every α , and so $\lambda(T) = 0$.

Claim 5. *For every non-separable $X_0 \hookrightarrow X$ every $\varepsilon > 0$ and every k there are two normalized vectors x and y such that $x \in X_0$, y is a $2 - \ell_1^k$ -average, $\|T(x)\| \leq \varepsilon$ and $\|x - y\| \leq \varepsilon$.*

It is easy to find the desired deciding pair (\mathbf{x}, \mathbf{y}) from a simple use of the previous claim. Now we pass to give a proof of the claim. Fix all data. Let $l \in \mathbb{N}$ be the result of the application of Proposition 2.7 to our fixed k , and let $\delta = \varepsilon/2l$. Since T is ω_1 -singular, one can find two normalized uncountable sequences $(x_\alpha)_{\alpha < \omega_1}$ and $(y_\alpha)_{\alpha < \omega_1}$ such that

- (a) $x_\alpha \in X_0$ for every $\alpha < \omega_1$.
- (b) $\mathbf{y} = (y_\alpha)_{\alpha < \omega_1}$ is a block sequence, and there an uncountable block sequence $(f_\alpha)_{\alpha < \omega_1}$ in K biorthogonal to \mathbf{y} .
- (c) $\|x_\alpha - y_\alpha\|, \|T(x_\alpha)\| \leq \delta$.

It follows from Corollary 2.5 when applied to $(\text{supp } f_\alpha)_{\alpha < \omega_1}$ that there is $F \subseteq \omega_1$ of size l such that $(f_\alpha)_{\alpha \in F}$ is a separated sequence. So, by Proposition 2.7 one can find a $2 - \ell_1^k$ in $\langle y_\alpha \rangle_{\alpha \in F}$ and then, using property (c) above, the counterpart $x \in \langle x_\alpha \rangle_{\alpha \in F}$ so that x and y fulfills the desired conditions. \square

From Theorem 3.13 one easily gets the main conclusion of this section which gives us the description of the spaces of operators $\mathcal{L}(X, \mathfrak{X})$ where X is an arbitrary closed infinite dimensional subspace of \mathfrak{X} .

Corollary 3.14. *Every bounded operator $T : X \rightarrow \mathfrak{X}$ from a closed subspace of \mathfrak{X} into \mathfrak{X} can be expressed as the sum*

$$T = \lambda(T)i_{X, \mathfrak{X}} + S$$

where S is a ω_1 -singular operator. \square

Corollary 3.15. *The space \mathfrak{X} is ω_1 -hereditarily indecomposable, and therefore it contains no uncountable unconditional basic sequences.*

PROOF. This follows from Proposition 3.7. \square

It is well known that the class of strictly singular operators on a Banach space X is a closed ideal of the Banach algebra $\mathcal{L}(X)$ of all bounded operators from X into X . We do not know if the same is true for the class of ω_1 -singular operators. However, we have the following.

Corollary 3.16. *If X is a closed infinite dimensional subspace of \mathfrak{X} , then the family $\mathcal{S}_{\omega_1}(X)$ of ω_1 -singular operators in $\mathcal{L}(X)$ forms a closed ideal in the Banach algebra $\mathcal{L}(X)$.*

PROOF. We show that $\lambda : \mathcal{L}(X) \rightarrow \mathbb{R}$, formally defined by $\lambda \mapsto \lambda(T) = \lambda(i_{X, \mathfrak{X}} \circ T)$, is a bounded operator between Banach algebras. From this one easily gets the desired result. Observe that we did almost all the work in Theorem 3.13. It remains to show that $\lambda(T_1 \circ T_0) = \lambda(T_1)\lambda(T_0)$. Fix a deciding pair (x, y) for both T_0 and T_1 . It follows that for all but countably many $\alpha < \omega_1$ one has that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|(T_1 \circ T_0)(x_k^\alpha) - \lambda(T_1 \circ T_0)x_k^\alpha\| &= 0 \\ \lim_{k \rightarrow \infty} \|(T_1 \circ T_0)(x_k^\alpha) - T_1(\lambda(T_0)x_k^\alpha)\| &= 0 \\ \lim_{k \rightarrow \infty} \|T_1(\lambda(T_0)x_k^\alpha) - \lambda(T_1)\lambda(T_0)x_k^\alpha\| &= 0, \end{aligned}$$

and so $\lambda(T_1 \circ T_0) = \lambda(T_1)\lambda(T_0)$, as desired. \square

In the case of $X = \mathfrak{X}$ we obtain the following slightly more informative result.

Corollary 3.17. *If $T : \mathfrak{X} \rightarrow \mathfrak{X}$ is a bounded operator, then $T(e_\alpha) = \lambda(T)e_\alpha$ for all but countably many α . It follows that T is the sum of a multiple of the identity plus an operator with separable range.*

PROOF. Otherwise we can find an uncountable subset $A \subseteq \omega_1$ and $\varepsilon > 0$ such that

$$\|T(e_\alpha) - \lambda(T)e_\alpha\| > \varepsilon \tag{13}$$

for all $\alpha \in A$ and also, as a consequence of Remark 2.2 (f), such that $T(e_\alpha) < T(e_\beta)$ for every $\alpha < \beta$ in A (here we are making an abuse of language by formally accepting that $0 < x$ for every vector x). Let $\theta_A : \omega \times \omega_1 \rightarrow A$ be the unique order-preserving onto mapping, and define $x_k^\alpha = e_{\theta_A(k, \alpha)}$. Then $((x_k^\alpha), (x_k^\alpha))$ is a deciding pair for T . Observe that this is what it makes so peculiar the situation $X = \mathfrak{X}$. So, by Lemma 3.9, for all but countably many α one has that

$$\lim_{k \rightarrow \infty} \|T(x_k^\alpha) - \lambda(T)x_k^\alpha\| = 0,$$

which is contradictory with (13). \square

The following result shows that, among the bounded operators from \mathfrak{X} into \mathfrak{X} , there is no distinction between the notions of ω_1 -singular operators and the notion of operators with separable range.

Corollary 3.18. *The following are equivalent for $T \in \mathcal{L}(\mathfrak{X})$:*

- (a) $(T(e_\alpha))_{\alpha < \omega_1}$ is eventually zero.
- (b) T has separable range.
- (c) T is ω_1 -singular.

PROOF. (c) implies (a): Suppose that T is ω_1 -singular. Then by Theorem 3.13 one has that $\lambda(T) = 0$; so by Corollary 3.17 it follows that $T(e_\alpha) = 0$ eventually. \square

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UNIVERSITÉ PARIS 7-CNRS, UMR 7056, 2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, CANADA, M5S 3G3